Module 6: Two Dimensional Problems in Polar Coordinate System

6.1.1 INTRODUCTION

In any elasticity problem the proper choice of the co-ordinate system is extremely important since this choice establishes the complexity of the mathematical expressions employed to satisfy the field equations and the boundary conditions.

In order to solve two dimensional elasticity problems by employing a polar co-ordinate reference frame, the equations of equilibrium, the definition of Airy's Stress function, and one of the stress equations of compatibility must be established in terms of Polar Co-ordinates.

6.1.2 STRAIN-DISPLACEMENT RELATIONS

Case 1: For Two Dimensional State of Stress

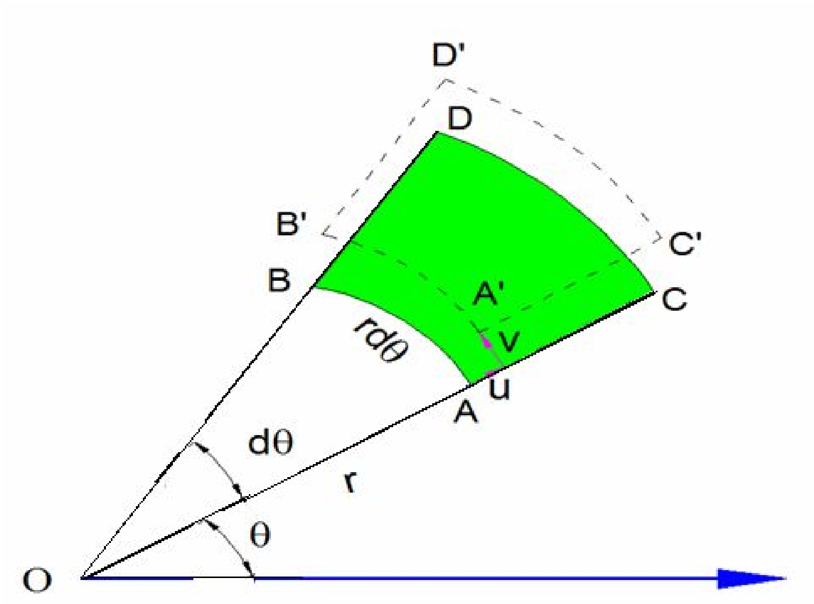

Figure 6.1 Deformed element in two dimensions

Consider the deformation of the infinitesimal element *ABCD*, denoting r and θ displacements by u and v respectively. The general deformation experienced by an element may be regarded as composed of (1) a change in the length of the sides, and (2) rotation of the sides as shown in the figure 6.1.

Referring to the figure, it is observed that a displacement *"u"* of side *AB* results in both radial and tangential strain.

Therefore, Radial strain = $\varepsilon_r = \frac{\partial u}{\partial r}$ (6.1)

and tangential strain due to displacement *u* per unit length of *AB* is

$$(\varepsilon_\theta)_u = \frac{(r+u)d\theta - rd\theta}{rd\theta} = \frac{u}{r} \quad (6.2)$$

Tangential strain due to displacement *v* is given by

$$(\varepsilon_\theta)_v = \frac{\left(\frac{\partial v}{\partial \theta}\right)d\theta}{rd\theta} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad (6.3)$$

Hence, the resultant strain is

$$\varepsilon_\theta = (\varepsilon_\theta)_u + (\varepsilon_\theta)_v$$

$$\varepsilon_\theta = \frac{u}{r} + \frac{1}{r}\left(\frac{\partial v}{\partial \theta}\right) \quad (6.4)$$

Similarly, the shearing strains can be calculated due to displacements *u* and *v* as below.

Component of shearing strain due to *u* is

$$(\gamma_{r\theta})_u = \frac{\left(\frac{\partial u}{\partial \theta}\right)d\theta}{rd\theta} = \frac{1}{r}\left(\frac{\partial u}{\partial \theta}\right) \quad (6.5)$$

Component of shearing strain due to *v* is

$$(\gamma_{r\theta})_v = \frac{\partial v}{\partial r} - \left(\frac{v}{r}\right) \quad (6.6)$$

Therefore, the total shear strain is given by

$$\gamma_{r\theta} = (\gamma_{r\theta})_u + (\gamma_{r\theta})_v$$

$$\gamma_{r\theta} = \frac{1}{r}\left(\frac{\partial u}{\partial \theta}\right) + \frac{\partial v}{\partial r} - \left(\frac{v}{r}\right) \tag{6.7}$$

Case 2: For Three -Dimensional State of Stress

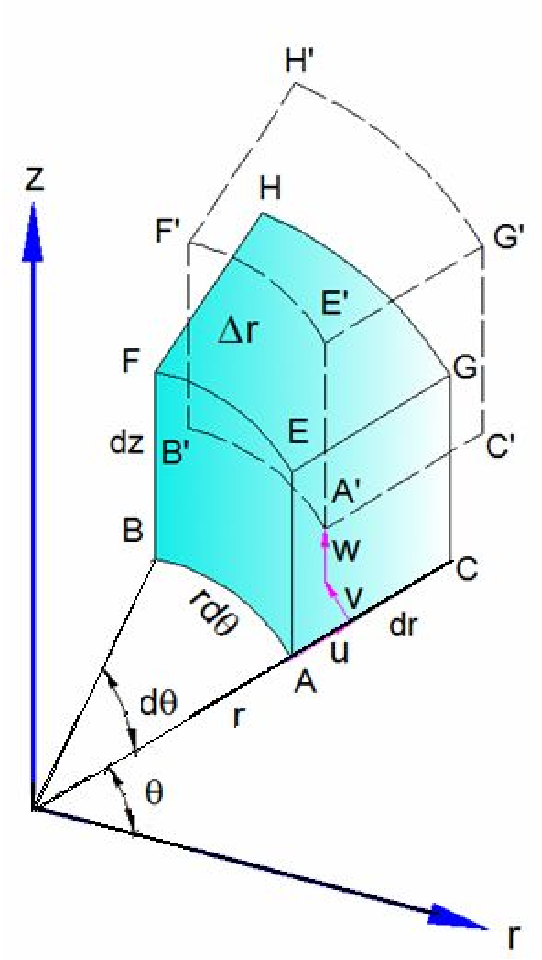

Figure 6.2 Deformed element in three dimensions

The strain-displacement relations for the most general state of stress are given by

$$\varepsilon_r = \frac{\partial u}{\partial r}, \quad \varepsilon_\theta = \frac{1}{r}\left(\frac{\partial v}{\partial \theta}\right) + \left(\frac{u}{r}\right), \quad \varepsilon_z = \frac{\partial w}{\partial z}$$

$$\gamma_{r\theta} = \frac{\partial v}{\partial r} + \frac{1}{r}\left(\frac{\partial u}{\partial \theta}\right) - \left(\frac{v}{r}\right) \tag{6.8}$$

$$\gamma_{\theta z} = \frac{1}{r}\left(\frac{\partial w}{\partial \theta}\right) + \left(\frac{\partial v}{\partial z}\right)$$

$$\gamma_{zr} = \frac{\partial u}{\partial z} + \left(\frac{\partial w}{\partial r}\right)$$

6.1.3 COMPATIBILITY EQUATION

We have from the strain displacement relations:

Radial strain, $\varepsilon_r = \dfrac{\partial u}{\partial r}$ (6.9a)

Tangential strain, $\varepsilon_\theta = \left(\dfrac{1}{r}\right)\dfrac{\partial v}{\partial \theta} + \left(\dfrac{u}{r}\right)$ (6.9b)

and total shearing strain, $\gamma_{r\theta} = \dfrac{\partial v}{\partial r} - \left(\dfrac{v}{r}\right) + \left(\dfrac{1}{r}\right)\dfrac{\partial u}{\partial \theta}$ (6.9c)

Differentiating Equation (6.9a) with respect to θ and Equation (6.9b) with respect to r, we get

$$\frac{\partial \varepsilon_r}{\partial \theta} = \frac{\partial^2 u}{\partial r \partial \theta} \qquad (6.9d)$$

$$\frac{\partial \varepsilon_\theta}{\partial r} = \left(\frac{1}{r}\right)\frac{\partial u}{\partial r} - \left(\frac{1}{r^2}\right)u + \frac{1}{r}.\frac{\partial^2 v}{\partial r \partial \theta} - \left(\frac{1}{r^2}\right).\frac{\partial v}{\partial \theta}$$

$$= \frac{\varepsilon_r}{r} + \left(\frac{1}{r}\right)\frac{\partial^2 v}{\partial r \partial \theta} - \frac{1}{r}\left[\frac{u}{r} + \left(\frac{1}{r}\right)\frac{\partial v}{\partial \theta}\right]$$

$$\therefore \frac{\partial \varepsilon_\theta}{\partial r} = \frac{\varepsilon_r}{r} + \left(\frac{1}{r}\right).\frac{\partial^2 v}{\partial r \partial \theta} - \left(\frac{1}{r}\right)\varepsilon_\theta \qquad (6.9e)$$

Now, Differentiating Equation (6.9c) with respect to r and using Equation (6.9d), we get

$$\frac{\partial \gamma_{r\theta}}{\partial r} = \frac{\partial^2 v}{\partial r^2} - \left(\frac{1}{r}\right)\frac{\partial v}{\partial r} + \frac{v}{r^2} + \left(\frac{1}{r}\right)\frac{\partial^2 u}{\partial r \partial \theta} - \left(\frac{1}{r^2}\right)\frac{\partial u}{\partial \theta}$$

$$= \frac{\partial^2 v}{\partial r^2} - \frac{1}{r}\left(\frac{\partial v}{\partial r} - \frac{v}{r} + \frac{1}{r}\frac{\partial u}{\partial \theta}\right) + \frac{1}{r}\frac{\partial^2 u}{\partial r \partial \theta}$$

$$\therefore \frac{\partial \gamma_{r\theta}}{\partial r} = \frac{\partial^2 v}{\partial r^2} - \left(\frac{1}{r}\right)\gamma_{r\theta} + \left(\frac{1}{r}\right)\frac{\partial \varepsilon_r}{\partial \theta} \qquad (6.9f)$$

Differentiating Equation (6.9e) with respect to r and Equation (6.9f) with respect to θ, we get,

$$\frac{\partial^2 \varepsilon_\theta}{\partial r^2} = \left(\frac{1}{r}\right)\frac{\partial \varepsilon_r}{\partial r} - \left(\frac{1}{r^2}\right)\varepsilon_r + \left(\frac{1}{r}\right)\frac{\partial^3 v}{\partial r^2 \partial \theta} - \left(\frac{1}{r^2}\right)\frac{\partial^2 v}{\partial r \partial \theta} - -\left(\frac{1}{r}\right)\frac{\partial \varepsilon_\theta}{\partial r} + \frac{1}{r^2}\varepsilon_\theta \qquad (6.9g)$$

and $\dfrac{\partial^2\gamma_{r\theta}}{\partial r\partial\theta} = \dfrac{\partial^3 v}{\partial r^2\partial\theta} - \left(\dfrac{1}{r}\right)\dfrac{\partial\gamma_{r\theta}}{\partial\theta} + \left(\dfrac{1}{r}\right)\dfrac{\partial^2\varepsilon_r}{\partial\theta^2}$

or $\left(\dfrac{1}{r}\right)\dfrac{\partial^2\gamma_{r\theta}}{\partial r\partial\theta} = \left(\dfrac{1}{r}\right)\dfrac{\partial^3 v}{\partial r^2\partial\theta} - \left(\dfrac{1}{r^2}\right)\dfrac{\partial\gamma_{r\theta}}{\partial\theta} + \left(\dfrac{1}{r^2}\right)\dfrac{\partial^2\varepsilon_r}{\partial\theta^2}$ (6.9h)

Subtracting Equation (6.9h) from Equation (6.9g) and using Equation (6.9e), we get,

$$\frac{\partial^2\varepsilon_\theta}{\partial r^2} - \left(\frac{1}{r}\right)\frac{\partial^2\gamma_{r\theta}}{\partial r\partial\theta} = \left(\frac{1}{r}\right)\frac{\partial\varepsilon_r}{\partial r} - \left(\frac{\varepsilon_r}{r^2}\right) - \left(\frac{1}{r^2}\right)\frac{\partial^2 v}{\partial r\partial\theta} - \left(\frac{1}{r}\right)\frac{\partial\varepsilon_\theta}{\partial r} + \left(\frac{1}{r^2}\right)\frac{\partial\gamma_{r\theta}}{\partial\theta} - \left(\frac{1}{r^2}\right)\frac{\partial^2\varepsilon_r}{\partial\theta^2} + \frac{\varepsilon_\theta}{r^2}$$

$$= \frac{1}{r}\left(\frac{\partial\varepsilon_r}{\partial r}\right) - \frac{1}{r}\left(\frac{\varepsilon_r}{r} + \frac{1}{r}\frac{\partial^2 v}{\partial r\partial\theta} - \frac{\varepsilon_\theta}{r}\right) - \frac{1}{r}\left(\frac{\partial\varepsilon_\theta}{\partial r} - \frac{1}{r}.\frac{\partial\gamma_{r\theta}}{\partial\theta} + \frac{1}{r}\frac{\partial^2\varepsilon_r}{\partial\theta^2}\right)$$

$$= \left(\frac{1}{r}\right)\frac{\partial\varepsilon_r}{\partial r} - \left(\frac{1}{r}\right)\frac{\partial\varepsilon_\theta}{\partial r} - \left(\frac{1}{r}\right)\frac{\partial\varepsilon_\theta}{\partial r} + \left(\frac{1}{r^2}\right)\frac{\partial\gamma_{r\theta}}{\partial\theta} - \left(\frac{1}{r^2}\right)\frac{\partial^2\varepsilon_r}{\partial\theta^2}$$

$$= \left(\frac{1}{r}\right)\frac{\partial\varepsilon_r}{\partial r} - \left(\frac{2}{r}\right)\frac{\partial\varepsilon_\theta}{\partial r} + \left(\frac{1}{r^2}\right)\frac{\partial\gamma_{r\theta}}{\partial\theta} - \left(\frac{1}{r^2}\right)\frac{\partial^2\varepsilon_r}{\partial\theta^2}$$

$$\therefore \left(\frac{1}{r^2}\right)\frac{\partial\gamma_{r\theta}}{\partial\theta} + \left(\frac{1}{r}\right)\frac{\partial^2\gamma_{r\theta}}{\partial r\partial\theta} = \frac{\partial^2\varepsilon_\theta}{\partial r^2} + \left(\frac{2}{r}\right)\frac{\partial\varepsilon_\theta}{\partial r} - \left(\frac{1}{r}\right)\frac{\partial\varepsilon_r}{\partial r} + \left(\frac{1}{r^2}\right)\frac{\partial^2\varepsilon_r}{\partial\theta^2}$$

6.1.4 STRESS-STRAIN RELATIONS

In terms of cylindrical coordinates, the stress-strain relations for 3-dimensional state of stress and strain are given by

$$\varepsilon_r = \frac{1}{E}[\sigma_r - \nu(\sigma_\theta + \sigma_z)]$$

$$\varepsilon_\theta = \frac{1}{E}[\sigma_\theta - \nu(\sigma_r + \sigma_z)] \qquad (6.10)$$

$$\varepsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_r + \sigma_\theta)]$$

For two-dimensional state of stresses and strains, the above equations reduce to,

For Plane Stress Case

$$\varepsilon_r = \frac{1}{E}(\sigma_r - \nu\sigma_\theta)$$

$$\varepsilon_\theta = \frac{1}{E}(\sigma_\theta - \nu\sigma_r) \qquad (6.11)$$

$$\gamma_{r\theta} = \frac{1}{G}\tau_{r\theta}$$

For Plane Strain Case

$$\varepsilon_r = \frac{(1+\nu)}{E}[(1-\nu)\sigma_r - \gamma\sigma_\theta]$$

$$\varepsilon_\theta = \frac{(1+\nu)}{E}[(1-\nu)\sigma_\theta - \gamma\sigma_r] \qquad (6.12)$$

$$\gamma_{r\theta} = \frac{1}{G}\tau_{r\theta}$$

6.1.5 AIRY'S STRESS FUNCTION

With reference to the two-dimensional equations or stress transformation [Equations (2.12a) to (2.12c)], the relationship between the polar stress components σ_r, σ_θ and $\tau_{r\theta}$ and the Cartesian stress components σ_x, σ_y and τ_{xy} can be obtained as below.

$$\begin{aligned}
\sigma_r &= \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + \tau_{xy}\sin 2\theta \\
\sigma_\theta &= \sigma_y \cos^2\theta + \sigma_x \sin^2\theta - \tau_{xy}\sin 2\theta \\
\tau_{r\theta} &= (\sigma_y - \sigma_x)\sin\theta\cos\theta + \tau_{xy}\cos 2\theta
\end{aligned} \qquad (6.13)$$

Now we have,

$$\sigma_x = \frac{\partial^2\phi}{\partial y^2} \qquad \sigma_y = \frac{\partial^2\phi}{\partial x^2} \qquad \tau_{xy} = -\frac{\partial^2\phi}{\partial x \partial y} \qquad (6.14)$$

Substituting (6.14) in (6.13), we get

$$\sigma_r = \frac{\partial^2\phi}{\partial y^2}\cos^2\theta + \frac{\partial^2\phi}{\partial x^2}\sin^2\theta - \frac{\partial^2\phi}{\partial x\partial y}\sin 2\theta$$

$$\sigma_\theta = \frac{\partial^2\phi}{\partial x^2}\cos^2\theta + \frac{\partial^2\phi}{\partial y^2}\sin^2\theta + \frac{\partial^2\phi}{\partial x\partial y}\sin 2\theta \qquad (6.15)$$

$$\tau_{r\theta} = \left(\frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\phi}{\partial y^2}\right)\sin\theta\cos\theta - \frac{\partial^2\phi}{\partial x\partial y}\cos 2\theta$$

The polar components of stress in terms of Airy's stress functions are as follows.

$$\sigma_r = \left(\frac{1}{r}\right)\frac{\partial\phi}{\partial r} + \left(\frac{1}{r^2}\right)\frac{\partial^2\phi}{\partial\theta^2} \qquad (6.16)$$

$$\sigma_\theta = \frac{\partial^2\phi}{\partial r^2} \text{ and } \tau_{r\theta} = \left(\frac{1}{r^2}\right)\frac{\partial\phi}{\partial\theta} - \left(\frac{1}{r}\right)\frac{\partial^2\phi}{\partial r\partial\theta} \qquad (6.17)$$

The above relations can be employed to determine the stress field as a function of r and θ.

6.1.6 BIHARMONIC EQUATION

As discussed earlier, the Airy's Stress function ϕ has to satisfy the Biharmonic equation $\nabla^4\phi = 0,$ provided the body forces are zero or constants. In Polar coordinates the stress function must satisfy this same equation; however, the definition of ∇^4 operator must be modified to suit the polar co-ordinate system. This modification may be accomplished by transforming the ∇^4 operator from the Cartesian system to the polar system.

Now, we have, $x = r\cos\theta, \quad y = r\sin\theta$

$$r^2 = x^2 + y^2 \text{ and } \theta = \tan^{-1}\left(\frac{y}{x}\right) \tag{6.18}$$

where r and θ are defined in Figure 6.3

Differentiating Equation (6.18) gives

$$\frac{\partial r}{\partial x} = \frac{x}{r} = \frac{r\cos\theta}{r} = \cos\theta$$

$$\frac{\partial r}{\partial y} = \frac{y}{r} = \frac{r\sin\theta}{r} = \sin\theta$$

$$\frac{\partial \theta}{\partial x} = -\left(\frac{y}{r^2}\right) = \frac{r\sin\theta}{r^2} = -\left(\frac{\sin\theta}{r}\right)$$

$$\frac{\partial \theta}{\partial y} = \frac{x}{r^2} = \frac{r\cos\theta}{r^2} = \frac{\cos\theta}{r}$$

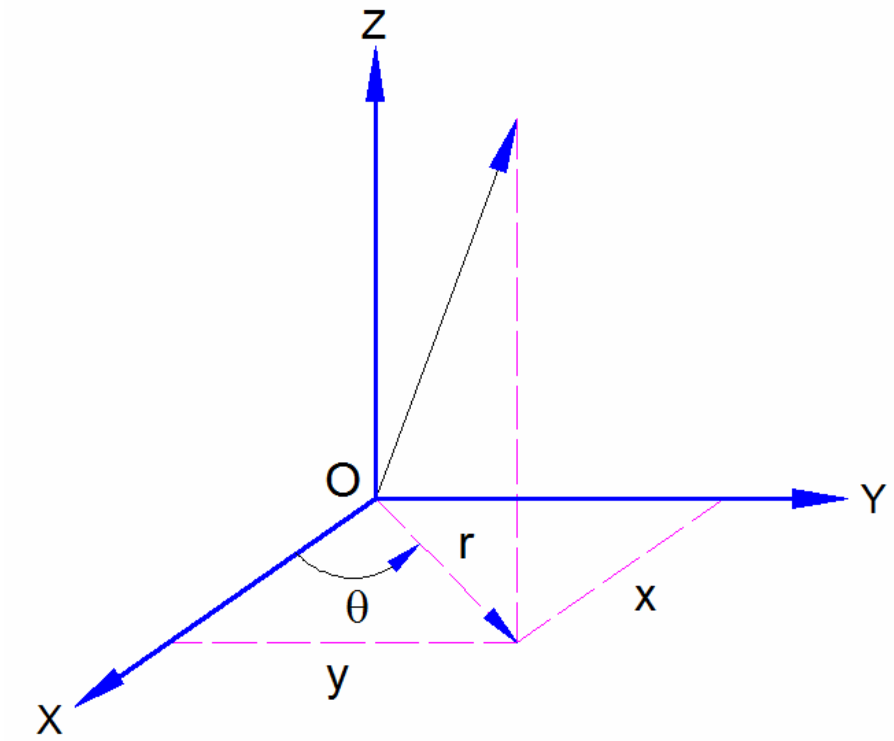

Figure.6.3

$$2r\,dr = 2x dx + 2y\,dy$$

$$\therefore dr = \left(\frac{x}{r}\right)dx + \left(\frac{y}{r}\right)dy$$

Also, $\sec^2\theta\; d\theta = -\left(\frac{y}{x^2}\right)xy + \left(\frac{dy}{x}\right)$

$$\frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial\phi}{\partial\theta}\frac{\partial\theta}{\partial x}$$

$$= \frac{x}{\sqrt{x^2+y^2}}\frac{\partial\phi}{\partial r} + \left(-\frac{1}{\sec^2\theta}\left(\frac{y}{x^2}\right)\right)\frac{\partial\phi}{\partial\theta}$$

$$\therefore \frac{\partial\phi}{\partial x} = \cos\theta\left(\frac{\partial\phi}{\partial r}\right) - \frac{\sin\theta}{r}\left(\frac{\partial\phi}{\partial\theta}\right)$$

Similarly, $\frac{\partial\phi}{\partial y} = \frac{\partial\phi}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial\phi}{\partial\theta}\frac{\partial\theta}{\partial y}$

$$\therefore \frac{\partial\phi}{\partial y} = \sin\theta\left(\frac{\partial\phi}{\partial r}\right) + \frac{\cos\theta}{r}\left(\frac{\partial\phi}{\partial\theta}\right)$$

Now, $\frac{\partial^2 \phi}{\partial x^2} = \left(\cos\theta \left(\frac{\partial \phi}{\partial r} \right) - \frac{\sin\theta}{r} \left(\frac{\partial \phi}{\partial \theta} \right) \right)^2$

$$= \cos^2\theta \frac{\partial^2 \phi}{\partial r^2} - \left(\frac{2\sin\theta\cos\theta}{r} \right) \frac{\partial^2 \phi}{\partial r \partial \theta} + \frac{\sin^2\theta}{r^2} \left(\frac{\partial^2 \phi}{\partial \theta^2} \right) + \frac{2\sin\theta\cos\theta}{r^2} \left(\frac{\partial \phi}{\partial \theta} \right) + \frac{\sin^2\theta}{r} \left(\frac{\partial \phi}{\partial r} \right) \quad \text{(i)}$$

Similarly,

$$\frac{\partial^2 \phi}{\partial y^2} = \sin^2\theta \frac{\partial^2 \phi}{\partial r^2} + \frac{2\sin\theta\cos\theta}{r} \frac{\partial^2 \phi}{\partial r \partial \theta} - \left(\frac{2\sin\theta\cos\theta}{r^2} \right) \frac{\partial \phi}{\partial \theta} + \frac{\cos^2\theta}{r} \left(\frac{\partial \phi}{\partial r} \right) + \frac{\cos^2\theta}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} \quad \text{(ii)}$$

And,

$$\frac{\partial^2 \phi}{\partial x \partial y} = -\left(\frac{\sin\theta\cos\theta}{r} \right) \frac{\partial \phi}{\partial r} + \sin\theta\cos\theta \frac{\partial^2 \phi}{\partial r^2} + \frac{\cos 2\theta}{r} \frac{\partial^2 \phi}{\partial r \partial \theta} - \left(\frac{\cos 2\theta}{r^2} \right) \frac{\partial \phi}{\partial \theta} - \left(\frac{\sin\theta\cos\theta}{r^2} \right) \frac{\partial^2 \phi}{\partial \theta^2} \quad \text{(iii)}$$

Adding (i) and (ii), we get

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{\partial^2 \phi}{\partial r^2} + \left(\frac{1}{r} \right) \frac{\partial \phi}{\partial r} + \left(\frac{1}{r^2} \right) \frac{\partial^2 \phi}{\partial \theta^2}$$

$$i.e, \nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{\partial^2 \phi}{\partial r^2} + \left(\frac{1}{r} \right) \frac{\partial \phi}{\partial r} + \left(\frac{1}{r^2} \right) \frac{\partial^2 \phi}{\partial \theta^2}$$

or $\nabla^4 \varphi = \nabla^2 (\nabla^2 \varphi) = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \left(\frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r} \frac{\partial \varphi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \varphi}{\partial \theta^2} \right) = 0$

The above Biharmonic equation is the stress equation of compatibility in terms of Airy's stress function referred in polar co-ordinate system.